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On a class of homogeneous cones consisting of real symmetric matrices

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Abstract. The cone of real positive definite symmetric matrices with prescribed zeros plays an important role in statistics. Letac and Massam [7] claimed a simple criterion for such a cone to be homogeneous. In this article, we give a complete proof of their statement and related results.

§1. Introduction.

The cone $\mathcal{P}_n$ of real positive definite symmetric matrices of size n is a fundamental object in statistics and various areas of mathematics, and fascinating analysis on $\mathcal{P}_n$ has been developed by numbers of authors. For instance, the so-called Siegel integral

$$\int_{\mathcal{P}_n} e^{-\operatorname{tr} xy} (\det x)^{\alpha-(r+1)/2}\, dx = \pi^{r(r-1)/4} \prod_{k=1}^{n} \Gamma(\alpha - \frac{k-1}{2}) (\det y)^{-\alpha} \qquad (y \in \mathcal{P}_n,\ \Re\alpha > \frac{r-1}{2}) \tag{1}$$

is first considered by Wishart [12] in the study of multivariate analysis, whereas Siegel [9] made good use of the integral in analytic number theory. The richness of the analysis on the cone $\mathcal{P}_n$ is due to a transitive action ρ of the group $GL(n,\mathbb{R})$ on the cone $\mathcal{P}_n$ given by $\rho(a)x := ax{}^{\mathrm{t}}a$ $(a \in GL(n,\mathbb{R}),\ x \in \mathcal{P}_n)$. Abstracting a transitive linear action on an open convex cone from this particular example, Vinberg [11] and Gindikin [1] established a basic theory of homogeneous cones, where the integral formula (1) is generalized to each homogeneous cone.

In statistics, the space of real symmetric matrices with prescribed zeros such as

$$\left\{ \begin{pmatrix} x_{11} & x_{12} & 0 & x_{14} \\ x_{12} & x_{22} & x_{23} & 0 \\ 0 & x_{23} & x_{33} & x_{34} \\ x_{14} & 0 & x_{34} & x_{44} \end{pmatrix} ;\ x_{ij} \in \mathbb{R} \right\} \tag{2}$$

and

$$\left\{ \begin{pmatrix} x_{11} & x_{12} & 0 & 0 \\ x_{12} & x_{22} & x_{23} & 0 \\ 0 & x_{23} & x_{33} & x_{34} \\ 0 & 0 & x_{34} & x_{44} \end{pmatrix} ;\ x_{ij} \in \mathbb{R} \right\} \tag{3}$$

and its subset consisting of positive definite matrices are quite important in view of the study of the covariance matrix of a random vector with a given conditional independence. The difficulty to handle this kind of matrices depends very much on the pattern of zeros, which is expressed by an undirected graph. Indeed, since the corresponding graph is chordal for the case (3) as we see below, there is an elaborate theory [6] to treat symmetric matrices with the particular zero pattern, and an integral formula similar to (1) holds ([7, 8]). On the other hand, there is no such a concise tool for the case (2). Letac and Massam [7] gave a simple condition of a graph for which the corresponding set of positive definite matrices forms a homogeneous cone. Actually, one can see that Vinberg-Gindikin theory and theory of chordal graph give the same result for the homogeneous cone arising from the graph. In [7], the condition is claimed as a part of a theorem without proof ([7, Theorem 2.2]). The aim of the present paper is to give a complete proof for their statement.

Let us explain the contents of this work in more detail. First we recall the definition of homogeneous cone. Let $\mathcal{Z}$ be a real vector space, and $\Omega \subset \mathcal{Z}$ an open convex cone containing no straight line. We denote by $GL(\Omega)$ the linear automorphism group $\{ g \in GL(\mathcal{Z})\,;\, g\Omega = \Omega \}$ of the cone Ω. The cone Ω is said to be *homogeneous* if $GL(\Omega)$ acts on Ω transitively. For example, put

$$\mathcal{Z} := \left\{ \begin{pmatrix} x_{11} & x_{12} & 0 \\ x_{12} & x_{22} & x_{23} \\ 0 & x_{23} & x_{33} \end{pmatrix} ; x_{ij} \in \mathbb{R} \right\}, \quad \Omega := \mathcal{Z} \cap \mathcal{P}_3. \tag{4}$$

Then Ω is a homogeneous cone. Indeed, the group $GL(\Omega)$ is generated by linear maps $\rho(a) : x \mapsto a x {}^{\mathrm{t}}a$ with $a = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{32} \\ 0 & 0 & a_{33} \end{pmatrix} \in GL(3,\mathbb{R})$ and $\rho(\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix})$, while we can verify that $GL(\Omega)$ acts on Ω transitively by applying Theorem 3.

Next we explain some notions about a graph. Let G be a graph, and V the set of vertices of G. We assume that G has no multiple edge, that is, for any two vertices $i, j \in V$, either there is one edge connecting them, or there is no edge between them. These relations of the vertices i and j are denoted by $i \sim j$ and $i \not\sim j$ respectively. Assume further that G has no loop, which means that $i \not\sim i$ for $i \in V$. We define the set $E \subset V \times V$ by

$$E := \{ (i,i)\,;\, i \in V \} \sqcup \{ (i,j) \in V \times V\,;\, i \sim j \}.$$

Since V and E have all information of G, the graph G is often identified with the pair (V, E). For a non-empty subset V' of V, put $E' := E \cap (V' \times V')$. The graph $G' := (V', E')$ is called an *induced subgraph* of G. The graph G is said to be *chordal* or *decomposable* if G contains no cycle of length greater than 3 as an induced subgraph, and said to be A_4*-free* if G contains no A_4 graph $\bullet - \bullet - \bullet - \bullet$ as an induced subgraph.

Let us label the vertices of G as $V = \{1, 2, \dots, n\}$, and define

$$\mathcal{Z}_G := \{ (x_{ij}) \in \mathrm{Sym}(n,\mathbb{R})\,;\, x_{ij} = 0 \text{ if } (i,j) \notin E \},$$
$$\mathcal{P}_G := \{ x \in \mathcal{Z}_G\,;\, x \text{ is positive definite} \}.$$

Then $\mathcal{Z}_G$ is a vector subspace of $\mathrm{Sym}(n,\mathbb{R})$, and $\mathcal{P}_G$ is an open convex cone in $\mathcal{Z}_G$. The spaces (2) and (3) are nothing but $\mathcal{Z}_G$ with G being the cycle of length 4 and the A_4 graph respectively. Now we state the Letac-Massam criterion.

Theorem A. *The cone $\mathcal{P}_G \subset \mathcal{Z}_G$ is homogeneous if and only if G is chordal and A_4-free.*

Let us explain the organization of the paper, giving the sketch of the proof of Theorem A. In Section 2, we show the 'if' part by constructing a linear group acting transitively on $\mathcal{P}_G$ from the graph G (Theorem 3). In Section 3, we show that

Theorem B (Theorem 7)**.** *If $\mathcal{P}_G \subset \mathcal{Z}_G$ is a homogeneous cone, the cone $\mathcal{P}_{G'} \subset \mathcal{Z}_{G'}$ is also homogeneous for any induced subgraph G' of G.*

Thanks to Theorem B, the proof of the 'only if' part of Theorem A is reduced to showing the non-homogeneity of the cones in the spaces (2) and (3) because the condition that G is chordal and A_4-free is equivalent to that G does not contain any cycle of length 4 nor A_4 graph as an induced subgraph. Our proofs of the non-homogeneity in Sections 4 and 5 as well as Theorem B are not entirely elementary. They require results about structures of left-symmetric algebra introduced by Vinberg [11], which is reviewed in Section 3.

For the reader's convenience, we describe the correspondence between our results and the original statement by Letac and Massam. In [7, Theorem 2.2], five properties about a connected undirected graph are claimed to be equivalent. Our Theorem A states the equivalence of the first and the fourth. Proposition 5 in Section 2 means the equivalence of the first and the second, while the succeeding argument, proof of Theorem 3, essentially shows that the second implies the fourth.

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§2. Construction of linear automorphisms on $\mathcal{P}_G$.

For $i \in V$, define the *neighborhood* $\mathrm{nb}(i) \subset V$ of i by $\mathrm{nb}(i) := \{\, j \in V\,;\, j \sim i \,\}$. Here we follow a conventional terminology, though the set $\{i\} \cup \mathrm{nb}(i)$ seems more natural to be called 'neighborhood'. We define

$$\mathcal{M}_G := \{\, (m_{ij}) \in \mathrm{Mat}(n, \mathbb{R})\,;\, m_{ij} = 0 \text{ if } (i,j) \notin E \,\}\,,$$
$$\mathcal{A}_G := \{\, a \in \mathcal{M}_G\,;\, a_{ij} = 0 \text{ if } \{i\} \cup \mathrm{nb}(i) \not\supset \{j\} \cup \mathrm{nb}(j) \,\}\,.$$

Since $(i,j) \in E$ if and only if $(j,i) \in E$, we see that

$$m \in \mathcal{M}_G \quad \text{if and only if} \quad {}^{\mathsf{t}}m \in \mathcal{M}_G. \tag{5}$$

Clearly, $\mathcal{Z}_G = \mathcal{M}_G \cap \mathrm{Sym}(n, \mathbb{R})$.

Lemma 1. *For $a \in \mathcal{A}_G$ and $b \in \mathcal{M}_G$, the matrix $c := ab$ belongs to $\mathcal{M}_G$. Moreover, c belongs to $\mathcal{A}_G$ if $b \in \mathcal{A}_G$.*

Proof. If the (i,j)-component $c_{ij} = \sum_{k \in V} a_{ik} b_{kj}$ is non-zero, there exists k for which $a_{ik} b_{kj} \neq 0$. Since $b_{kj} \neq 0$ and $b \in \mathcal{M}_G$, we have $(k,j) \in E$ so that $j \in \{k\} \cup \mathrm{nb}(k)$. On the other hand, we see from $a \in \mathcal{A}_G$ and $a_{ik} \neq 0$ that $\{i\} \cup \mathrm{nb}(i) \supset \{k\} \cup \mathrm{nb}(k)$. Thus $j \in \{i\} \cup \mathrm{nb}(i)$. Namely $c_{ij} \neq 0$ implies $(i, j) \in E$, which means that $c \in \mathcal{M}_G$.

If $b \in \mathcal{A}_G$, then $b_{kj} \neq 0$ implies $\{k\} \cup \mathrm{nb}(k) \supset \{j\} \cup \mathrm{nb}(j)$. Thus $c_{ij} \neq 0$ implies $\{i\} \cup \mathrm{nb}(i) \supset \{j\} \cup \mathrm{nb}(j)$, and we get $c \in \mathcal{A}_G$. □

Lemma 1 tells us that $\mathcal{A}_G$ is a subalgebra of the matrix algebra $\mathrm{Mat}(n, \mathbb{R})$, whereas $\mathcal{M}_G$ is not so in general. If $a \in \mathcal{A}_G$ is invertible, then a^{-1} belongs to $\mathcal{A}_G$ because a^{-1} is expressed as a polynomial of a by the Cayley-Hamilton theorem. Thus $\{\, a \in \mathcal{A}_G\, ;\, \det a \neq 0 \,\}$ forms a group, which we denote by $\mathcal{A}_G^\times$.

Lemma 2. *For $a \in \mathcal{A}_G^\times$ and $x \in \mathcal{Z}_G$, the matrix $\rho(a)x = ax{}^{\mathsf{t}}a$ belongs to $\mathcal{Z}_G$.*

Proof. Thanks to Lemma 1, we have $b := ax \in \mathcal{M}_G$. Then ${}^{\mathsf{t}}b \in \mathcal{M}_G$ by (5), and $a{}^{\mathsf{t}}b \in \mathcal{M}_G$ by Lemma 1. Using (5) again, we get $b{}^{\mathsf{t}}a = ax{}^{\mathsf{t}}a \in \mathcal{M}_G \cap \mathrm{Sym}(n, \mathbb{R}) = \mathcal{Z}_G$. □

Lemma 2 gives us the group homomorphism $\rho : \mathcal{A}_G^\times \to GL(\mathcal{P}_G)$. The aim of this section is to prove the following theorem:

Theorem 3. *If G is chordal and A_4-free, then $\rho(\mathcal{A}_G^\times)$ acts on $\mathcal{P}_G$ transitively. In particular, $\mathcal{P}_G$ is a homogeneous cone.*

Noting that the inclusion relation of the sets $\{i\} \cup \mathrm{nb}(i)$ defines a partial preorder on V, we introduce a partial order $\succeq$ on V in such a way that

(O1) $\{i\} \cup \mathrm{nb}(i) \supsetneq \{j\} \cup \mathrm{nb}(j)$ implies $i \succeq j$,

(O2) $\{i\} \cup \mathrm{nb}(i) = \{j\} \cup \mathrm{nb}(j)$ implies $i \succeq j$ or $j \succeq i$.

Such a partial order is not unique, whereas we fix one order $\succeq$. Define

$$\mathcal{T}_{\succeq} := \left\{ a \in \mathcal{A}_G\, ;\, \begin{array}{ll} a_{ii} > 0 & (i \in V) \\ a_{ij} = 0 & (i \not\succeq j) \end{array} \right\}.$$

Then $\mathcal{T}_{\succeq}$ is a subgroup of $\mathcal{A}_G^\times$.

In general, a linear group H on a vector space $\mathcal{Z}$ is said to be *triangularizable* if there exists a basis $\{e_1, \ldots, e_N\}$ of $\mathcal{Z}$ such that each element of H is expressed as a lower triangular matrix with respect to the basis.

Lemma 4. *The group $\rho(\mathcal{T}_{\succeq}) \subset GL(\mathcal{P}_G)$ is triangularizable, and acts on $\mathcal{P}_G$ freely.*

Proof. Let $\mathcal{T}_n$ be the group of real lower triangular matrices of size n with positive diagonals. Then it is well-known that $\rho(\mathcal{T}_n)$ is triangularizable and acts on $\mathcal{P}_n$ simply transitively. On the other hand, since $\succeq$ is a partial order on the finite set $V = \{1, 2, \ldots, n\}$, there exists a permutation

$\sigma \in \mathfrak{S}_n$ such that $i \succeq j$ implies $\sigma(i) \geq \sigma(j)$. Let $w_\sigma \in GL(n, \mathbb{R})$ be the permutation matrix corresponding to σ. Take $a \in \mathcal{T}_\succeq$ and put $a' := w_\sigma a w_\sigma^{-1} \in \mathrm{Mat}(n, \mathbb{R})$. Then the (i, j)-component a'_{ij} of a' equals $a_{\sigma^{-1}(i), \sigma^{-1}(j)}$. If $a'_{ij} = a_{\sigma^{-1}(i), \sigma^{-1}(j)} \neq 0$, then $\sigma^{-1}(i) \succeq \sigma^{-1}(i)$, which implies that $i \geq j$. Thus $a' \in \mathcal{T}_n$ and we conclude $w_\sigma \mathcal{T}_\succeq w_\sigma^{-1} \subset \mathcal{T}_n$, whence Lemma 4 follows easily. $\square$

As is mentioned in Introduction, the following statement is essentially a part of [7, Theorem 2.2] given without proof.

Proposition 5. *The graph G is chordal and A_4-free if and only if $\{i\} \cup \mathrm{nb}(i) \supset \{j\} \cup \mathrm{nb}(j)$ or $\{j\} \cup \mathrm{nb}(j) \supset \{i\} \cup \mathrm{nb}(i)$ holds for any pair $(i, j) \in E$.*

Proof. Assume that there exists $(i, j) \in E$ such that $\{i\} \cup \mathrm{nb}(i) \not\supset \{j\} \cup \mathrm{nb}(j)$ and $\{j\} \cup \mathrm{nb}(j) \not\supset \{i\} \cup \mathrm{nb}(i)$. Take $i_1 \in \{i\} \cup \mathrm{nb}(i) \setminus (\{j\} \cup \mathrm{nb}(j))$ and $j_1 \in \{j\} \cup \mathrm{nb}(j) \setminus (\{i\} \cup \mathrm{nb}(i))$. Then we have $i_1 \sim i \sim j \sim j_1$, $i_1 \not\sim j$, and $i \not\sim j_1$. Let us consider the induced subgraph $G' = (V', E')$ with $V' = \{i_1, i, j, j_1\}$. If $i_1 \sim j_1$, then G' is the cycle of length 4, so that G is not chordal. If $i_1 \not\sim j_1$, then G' is the A_4 graph, so that G is not A_4-free. Hence the 'if' part is proved. The 'only if' part is shown similarly. $\square$

Now we prove Theorem 3. Assume that G is chordal and A_4-free. Proposition 5 tells us that if $i \sim j$, then either $i \succeq j$ or $j \succeq i$ holds. Thus the dimension of $\mathcal{Z}_G$ is equal to the dimension of $\mathcal{T}_\succeq$ as a Lie group. Since $\rho(\mathcal{T}_\succeq)$ acts on $\mathcal{P}_G$ freely by Lemma 4, the $\rho(\mathcal{T}_\succeq)$-orbits in $\mathcal{P}_G$ are open. Therefore, owing to the connectedness of the convex cone $\mathcal{P}_G$, there must be only one $\rho(\mathcal{T}_\succeq)$-orbit in $\mathcal{P}_G$, which means that $\rho(\mathcal{T}_\succeq)$ acts on $\mathcal{P}_G$ (simply) transitively. Hence $\rho(\mathcal{A}_G^\times)$ acts on $\mathcal{P}_G$ transitively, too.

§3. Homogeneous cone and left symmetric algebra.

Let $\Omega \subset \mathcal{Z}$ be a homogeneous cone. Since the identity component $GL(\Omega)^\circ$ of the group $GL(\Omega)$ is equal to the identity component of an algebraic group, a maximal connected triangularizable subgroup H of $GL(\Omega)^\circ$ is unique up to inner isomorphisms, and H acts on Ω simply transitively ([10], [11, Chapter 1, Section 9]). The group $\rho(\mathcal{T}_\succeq)$ in Section 2 is an example of such an H. Taking a point $e \in \mathcal{Z}$, we have a diffeomorphism $H \ni g \mapsto ge \in \Omega$. Differentiating the diffeomorphism, we obtain a linear isomorphism $\mathfrak{h} \ni L \mapsto Le \in \mathcal{Z}$, where $\mathfrak{h} \subset End(\mathcal{Z})$ is the Lie algebra of H. For each $x \in \mathcal{Z}$, there exists a unique $L_x \in \mathfrak{h}$ for which $L_x e = x$. Let us define a bilinear product $\triangle : \mathcal{Z} \times \mathcal{Z} \to \mathcal{Z}$ by

$$x \triangle y := L_x y \in \mathcal{Z} \quad (x, y \in \mathcal{Z}).$$

Let $I_{\mathcal{Z}}$ denote the identity map on $\mathcal{Z}$. The one-parameter group of dilations $\{e^t I_{\mathcal{Z}}\}_{t\in\mathbb{R}}$ is contained in H, so that its generator $I_{\mathcal{Z}}$ belongs to $\mathfrak{h}$. Then we have $L_e = I_{\mathcal{Z}}$, which means that $e\triangle x = x$ for $x \in \mathcal{Z}$. On the other hand, $x\triangle e = L_x e = x$ by definition, so that $e \in \mathcal{Z}$ is a unit element of the algebra $(\mathcal{Z}, \triangle)$, Moreover $(\mathcal{Z}, \triangle)$ satisfies the following ([11, Chapter 2, Proposition 1]):
(Z1) $[x\triangle y\triangle z] = [y\triangle x\triangle z]$ for $x, y, z \in \mathcal{Z}$, where $[x\triangle y\triangle z] := x\triangle(y\triangle z)-(x\triangle y)\triangle z$ (left-symmetry),
(Z2) $(x|y)_{\mathrm{Tr}} := \mathrm{Tr}\, L_{x\triangle y}$ defines an inner product on $\mathcal{Z}$ (compactness),
(Z3) L_x has only real eigenvalues for each $x \in \mathcal{Z}$ (normality).
In general, an algebra $(\mathcal{Z}, \triangle)$ satisfying (Z1)–(Z3) is called a compact normal left-symmetric algebra (clan). Vinberg [11, Chapter 2, Theorem 2] showed that any clan with a unit element gives rise to a homogeneous cone, and the correspondence is one-to-one up to natural isomorphisms. Furthermore the clan $(\mathcal{Z}, \triangle)$ admits a kind of the Peirce decomposition (called a *normal decomposition*) as follows [11, Chapter 2, Proposition 8]: Let $\{e_1, \dots, e_r\}$ be a family of primitive idempotents of $\mathcal{Z}$ such that $e = e_1 + \cdots + e_r$. If $e_1, \dots, e_r$ are labeled appropriately, $\mathcal{Z}$ is decomposed as

$$\begin{aligned} &\mathcal{Z} = \sum_{1\le k<l\le r}^{\oplus} \mathcal{Z}_{lk}, \\ &\mathcal{Z}_{lk} := \{\, x \in \mathcal{Z}\,;\, e_i\triangle x = (\delta_{li} + \delta_{ki})x/2,\ x\triangle e_i = \delta_{ki}x \ \ (i = 1, \dots, r)\,\}\,. \end{aligned} \tag{6}$$

The space $\mathcal{Z}_{kk}$ is equal to $\mathbb{R}e_{kk}$ for $k = 1, \dots, r$, while other $\mathcal{Z}_{lk}$ $(l > k)$ can be $\{0\}$. The following multiplication rules hold:

$$\begin{gathered} \mathcal{Z}_{lk}\triangle\mathcal{Z}_{kj} \subset \mathcal{Z}_{lj}, \\ \text{if } k \ne i, j, \text{ then } \mathcal{Z}_{lk}\triangle\mathcal{Z}_{ij} = 0, \\ \mathcal{Z}_{lk}\triangle\mathcal{Z}_{mk} \subset \mathcal{Z}_{lm},\ \mathcal{Z}_{ml} \text{ according to } l \ge m \text{ or } m \ge l. \end{gathered} \tag{7}$$

For $I \subset \{1, \dots, r\}$, put $e_I := \sum_{i\in I} e_I \in \mathcal{Z}$. Then e_I is an idempotent, and any idempotent of $\mathcal{Z}$ is of this form ([11, Chapter 2, Proposition 9]). Let us consider the eigenspaces of the linear operator L_{e_I}. Define $\mathcal{Z}(L_{e_I}; \mu) := \{\, x \in \mathcal{Z}\,;\, e_I\triangle x = \mu x\,\}$ for $\mu \in \mathbb{R}$. Then we see from (6) that

$$\mathcal{Z} = \mathcal{Z}(L_{e_I}; 1) \oplus \mathcal{Z}(L_{e_I}; 1/2) \oplus \mathcal{Z}(L_{e_I}; 0) \tag{8}$$

with

$$\mathcal{Z}(L_{e_I}; 1) = \sum_{k,l\in I}^{\oplus} \mathcal{Z}_{lk}, \tag{9}$$

$$\mathcal{Z}(L_{e_I}; 0) = \sum_{k,l\notin I}^{\oplus} \mathcal{Z}_{lk}, \quad \mathcal{Z}(L_{e_I}; 1/2) = \sum_{k\in I, l\notin I \text{ or } k\notin I, l\in I}^{\oplus} \mathcal{Z}_{lk}. \tag{10}$$

Thanks to (9), we get a characterization for an idempotent to be primitive ([11, p.377, Corollary]).

Lemma 6. *The idempotent e_I of $\mathcal{Z}$ is primitive if and only if* $\dim \mathcal{Z}(L_{e_I}; 1) = 1$.

We see from (7) and (9) that the subspace $\mathcal{Z}_I := \mathcal{Z}(L_{e_I}; 1)$ forms a subalgebra of the clan $\mathcal{Z}$, where e_I is a unit element. Let $\pi_I : \mathcal{Z} \to \mathcal{Z}_I$ be the projection along the decomposition (8). Then the image $\pi_I(\Omega) \subset \mathcal{Z}_I$ is a homogeneous cone, which corresponds to the clan $\mathcal{Z}_I$ (cf. [3, Section 4]).

Now we assume that $\mathcal{P}_G \subset \mathcal{Z}_G$ is a homogeneous cone, and apply the argument above to this cone. For $\underline{c} := (c_1, \dots, c_n) \in \mathbb{R}^n$, we denote by $d_{\underline{c}}$ the diagonal matrix of size n whose (i,i)-component is c_i for $i = 1, \dots, n$. If $\underline{c} \in \mathbb{R}^n_{>0}$, then $\rho(d_{\underline{c}})$ belongs to $GL(\mathcal{P}_G)$. Let $H \subset GL(\mathcal{P}_G)^\circ$ be a maximal connected triangularizable subgroup containing $\{ \rho(d_{\underline{c}}) \,;\, \underline{c} \in \mathbb{R}^n_{>0} \}$. The Lie algebra $\mathfrak{h} \subset \mathrm{End}(\mathcal{Z}_G)$ contains linear maps

$$\dot{\rho}(d_{\underline{c}}) : \mathcal{Z}_G \ni x \mapsto d_{\underline{c}} x + x d_{\underline{c}} \in \mathcal{Z}_G$$

for $\underline{c} \in \mathbb{R}^n$. Putting $e := I_n$, we have $\dot{\rho}(d_{\underline{c}})e/2 = d_{\underline{c}}$, so that $L_{d_{\underline{c}}} = \dot{\rho}(d_{\underline{c}})/2$. Namely, we obtain

$$d_{\underline{c}} \triangle x = (d_{\underline{c}} x + x d_{\underline{c}})/2 \quad (x \in \mathcal{Z}_G). \tag{11}$$

In particular, we see that $d_{\underline{\varepsilon}}$ is an idempotent for $\underline{\varepsilon} \in \{0,1\}^r$. Note that the whole structure of $(\mathcal{Z}_G, \triangle)$ is not determined here, since the group H is not explicitly given. Nevertheless, we see from (11) that

$$\begin{aligned} \mathcal{Z}_G(L_{d_{\underline{\varepsilon}}}; 1) &= \{ x \in \mathcal{Z}_G \,;\, x_{ij} = 0 \text{ unless } \varepsilon_i = \varepsilon_j = 1 \}, \\ \mathcal{Z}_G(L_{d_{\underline{\varepsilon}}}; 0) &= \{ x \in \mathcal{Z}_G \,;\, x_{ij} = 0 \text{ unless } \varepsilon_i = \varepsilon_j = 0 \}, \\ \mathcal{Z}_G(L_{d_{\underline{\varepsilon}}}; 1/2) &= \{ x \in \mathcal{Z}_G \,;\, x_{ij} = 0 \text{ if } \varepsilon_i = \varepsilon_j = 0 \text{ or } \varepsilon_i = \varepsilon_j = 1 \}. \end{aligned} \tag{12}$$

The projection $\pi_{\underline{\varepsilon}} : \mathcal{Z}_G \ni x \to y \in \mathcal{Z}_G(L_{d_{\underline{\varepsilon}}}; 1)$ along the eigenspace decomposition of $\mathcal{Z}_G$ is given by

$$y_{ij} := \begin{cases} x_{ij} & (\varepsilon_i = \varepsilon_j = 1), \\ 0 & (\text{otherwise}). \end{cases}$$

Put $V' := \{ i \,;\, \varepsilon_i = 1 \}$, and let $G' = (V', E')$ be the induced subgraph of G. The space $\mathcal{Z}_G(L_{d_{\underline{\varepsilon}}}; 1)$ is naturally identified with $\mathcal{Z}_{G'}$ by $\mathcal{Z}_G(L_{d_{\underline{\varepsilon}}}; 1) \ni y \mapsto (y_{ij})_{i,j \in V'} \in \mathcal{Z}_{G'}$. Under this identification, $\pi_{\underline{\varepsilon}}(\mathcal{P}_G)$ equals $\mathcal{P}_{G'} \subset \mathcal{Z}_{G'}$. In conclusion, we obtain

Theorem 7. *If $\mathcal{P}_G \subset \mathcal{Z}_G$ is a homogeneous cone, the cone $\mathcal{P}_{G'} \subset \mathcal{Z}_{G'}$ is also homogeneous for any induced subgraph G' of G.*

In the rest of the section, we present some properties of a general homogeneous cone $\Omega \subset \mathcal{Z}$, and the normal decomposition (6) of $\mathcal{Z}$. Let $\mathcal{O}_1$ be the H-orbit in $\mathcal{Z}$ through the primitive idempotent e_1. Every element x of $\mathcal{O}_1$ is expressed as

$$x = t_{11}^2 e_1 + \sum_{m=2}^{r} t_{11} \tau_{m1} + \sum_{2 \le k \le m \le r} \tau_{m1} \triangle \tau_{k1} \tag{13}$$

with unique $t_{11} > 0$ and $\tau_{k1} \in \mathcal{Z}_{k1}$ $(k = 2, \dots, r)$ (cf. [3, Proposition 2.5], [4, (1.10)]). Since $e_1 \in \partial\Omega$, we have $\mathcal{O}_1 \subset \partial\Omega$. Moreover, we can deduce from (13) and [5, Section 3] that

$$\pi_I(\mathcal{O}_1) \subset \partial\Omega_I \quad (I = \{1, k\},\ k = 2, \dots, r). \tag{14}$$

Moreover, (13) tells us the following.

Lemma 8. *For $c_1 > 0$ and $u \in \mathcal{Z}(L_{e_1}; 1/2) = \sum^{\oplus}_{2\le m\le r} \mathcal{Z}_{m1}$, there exists a unique $y \in \mathcal{Z}(L_{e_1}; 0) = \sum^{\oplus}_{2\le k\le m\le r} \mathcal{Z}_{mk}$ such that $c_1 e_1 + u + y \in \mathcal{O}_1$.*

Put $e' := e - e_1 = e_2 + \cdots + e_r$. Then $\mathcal{Z}(L_{e_1}; 0) = \mathcal{Z}(L_{e'}; 1)$, which we denote by $\mathcal{Z}'$. Let $\Omega' \subset \mathcal{Z}'$ be the homogeneous cone corresponding to the subclan $\mathcal{Z}'$. Namely, Ω' is the H'-orbit through $e' \in \mathcal{Z}'$, where H' is a subgroup $\{\exp L_w\,;\, w \in \mathcal{Z}'\}$ of H.

Lemma 9. *For $u \in \mathcal{Z}(L_{e_1}; 1/2), y \in \mathcal{Z}'$ and $h \in H'$, one has $hu \in \mathcal{Z}(L_{e_1}; 1/2)$ and $hy \in \mathcal{Z}'$. Moreover $e_1 + u + y \in \mathcal{O}_1$ if and only if $e_1 + hu + hy \in \mathcal{O}_1$.*

Proof. Take $w \in \mathcal{Z}'$ for which $h = \exp L_w$. We see from (7) that $w \triangle u \in \mathcal{Z}(L_{e_1}; 1/2)$ and that $w \triangle e_1 = 0$. Thus $hu \in \mathcal{Z}(L_{e_1}; 1/2)$ and $he_1 = e_1$. Since $y \in \mathcal{Z}'$, it is clear that $hy \in \mathcal{Z}'$. Lemma 9 follows from this observation. □

§4. Non-homogeneity of the cone corresponding to the cycle of length 4.

Let $\mathcal{Z}_G$ be the vector space in (2), and suppose that $\mathcal{P}_G := \mathcal{Z}_G \cap \mathcal{P}_4$ is a homogeneous cone. Then $\mathcal{Z}_G$ becomes a clan in the way explained in Section 3. When $\underline{\varepsilon} = (0, \dots, \underset{i}{1}, \dots, 0) \in \mathbb{R}^n$, the matrix $d_{\underline{\varepsilon}}$ equals E_{ii}. Thus E_{ii} is an idempotent of $\mathcal{Z}_G$. Furthermore, Lemma 6 together with (12) tells us that E_{ii} is a primitive idempotent. Therefore, there exists a permutation matrix $\sigma \in \mathfrak{S}_4$ such that $E_{ii} = e_{\sigma(i)}$ for $i = 1, \dots, 4$. Let us assume that $E_{22} = e_1$. Applying Lemma 8, there exist unique $x_{11}, x_{14}, x_{33}, x_{34}, x_{44} \in \mathbb{R}$ for which

$$x = \begin{pmatrix} x_{11} & 1 & 0 & x_{14} \\ 1 & 1 & 1 & 0 \\ 0 & 1 & x_{33} & x_{34} \\ x_{14} & 0 & x_{34} & x_{44} \end{pmatrix} \in \mathcal{O}_1.$$

Thanks to (14), we obtain $x_{11} = 1$ and $x_{33} = 1$. Thus the third principal minor of x is

$$\begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = -1 < 0$$

which contradicts $x \in \mathcal{O}_1 \subset \partial\mathcal{P}_G$. The other assumption $E_{ii} = e_1$ with $i = 1, 3, 4$ also bring the contradiction in the same argument. Therefore we conclude that the cone $\mathcal{P}_G$ is not homogeneous.

§5. Non-homogeneity of the cone corresponding to the A_4 graph.

Let $\mathcal{Z}_G$ be the vector space in (3), and suppose that $\mathcal{P}_G := \mathcal{Z}_G \cap \mathcal{P}_4$ is a homogeneous cone, so that $\mathcal{Z}_G$ is a clan with primitive idempotents $E_{11}, \dots, E_{44}$. We shall search i for which $E_{ii} = e_1$.

In the same reason as in Section 4, we have $i \neq 2, 3$. Let us assume that $E_{11} = e_1$. By Lemma 8 and (14), we have

$$\mathcal{O}_1 = \left\{ \begin{pmatrix} c & u & 0 & 0 \\ u & u^2/c & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} ;\, c > 0,\, u \in \mathbb{R} \right\}. \tag{15}$$

Now let us observe the subspace

$$\mathcal{Z}' = \mathcal{Z}(L_{E_{11}}; 0) = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & x_{22} & x_{23} & 0 \\ 0 & x_{23} & x_{33} & x_{34} \\ 0 & 0 & x_{34} & x_{44} \end{pmatrix} ;\, x_{ij} \in \mathbb{R} \right\},$$

which is identified with

$$\left\{ \begin{pmatrix} x_{22} & x_{23} & 0 \\ x_{23} & x_{33} & x_{34} \\ 0 & x_{34} & x_{44} \end{pmatrix} ;\, x_{ij} \in \mathbb{R} \right\}.$$

The homogeneous cone $\Omega' \subset \mathcal{Z}'$ is nothing but the cone in (4). Combining (15) with Lemma 9, we see that H' must preserve the subspace

$$\mathcal{Z}'' := \left\{ \begin{pmatrix} x_{22} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} ;\, x_{22} \in \mathbb{R} \right\}.$$

On the other hand, recalling the description of $GL(\Omega')$ in Section 1, we obtain

$$H' \subset \{\, g \in GL(\Omega')\,;\, g\mathcal{Z}'' = \mathcal{Z}'' \,\} = \left\{ \rho(\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix})\,;\, a_{ij} \in \mathbb{R},\ a_{11}a_{22}a_{33} \neq 0 \right\}.$$

Therefore the dimension of H' is at most 4, which contradicts the fact that H' acts transitively on the cone Ω' of dimension 5. In the same argument, $E_{44} = e_1$ also causes a contradiction. Hence we conclude that $\mathcal{P}_G$ is not a homogeneous cone, and the proof of Theorem A is completed.

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